Math 347: Homework 7
Due on: Nov. 9, 2018

1. Sum of cubes.

   (i) Prove that
   $$m^3 = 6\binom{m}{3} + 6\binom{m}{2} + m;$$
   (ii) Use part (i) to prove that
   $$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2;$$
   (iii) Prove part (i) by a counting argument.

2. Determine the number of positive integer solutions to the equation
   $$x_1 + \cdots + x_k = N,$$
   for $k, N \in \mathbb{N}$.

   Brazil has 5 different types of coins, and the biggest city in the country is São Paulo with 12,106,920 people. Thus, for two people to have a change of having two different distributions of coins at home each person needs to have at least 57 coins. Answer the similar question for your country.

3. Prove that
   $$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}.$$

4. For a natural number n a *partition* of n is a way of writing n as a sum of positive numbers. Prove that the number of partitions of n into distinct numbers equals the number of partition of n into odd numbers.

5. The goal of this exercise is to determine which $p \in \mathbb{Q}[x]$ have the property that
   $$p(n) \in \mathbb{Z}, \text{ for all } n \in \mathbb{Z}.$$
   Let $I \subset \mathbb{Q}[x]$ be the subset of polynomials with this property.

   (i) Show that if $p, q \in I$ then $p + q \in I$ and $n \cdot p \in I$ for $n \in \mathbb{Z}$;
   (ii) Show that $p_j = \binom{x}{j}$ belongs to I, and more generally that for any collection of integers $n_j \in \mathbb{Z}$, for $0 \leq j \leq k$ one has
   $$\sum_{j=0}^{k} n_j p_j \in I;$$
   (iii) Let $f \in \mathbb{Q}[x]$ prove that f can be expressed as
   $$f = \sum_{j=0}^{k} a_j p_j,$$
   where $a_j \in \mathbb{Q}$;
   (iv) Prove that for $f \in \mathbb{Q}[x]$ one has
   $$f \in I \text{ if and only if } f = \textstyle\sum_{j=0}^{k} a_j p_j, \text{ with } a_j \in \mathbb{Z} \text{ for all } j.$$